

# LIPSCHITZ FUNCTIONS OF PERTURBED OPERATORS

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## Abstract.

We prove that if  $f$  is a Lipschitz function on  $\mathbb{R}$ ,  $A$  and  $B$  are self-adjoint operators such that  $\text{rank}(A - B) = 1$ , then  $f(A) - f(B)$  belongs to the weak space  $\mathbf{S}_{1,\infty}$ , i.e.,  $s_j(A - B) \leq \text{const}(1 + j)^{-1}$ . We deduce from this result that if  $A - B$  belongs to the trace class  $\mathbf{S}_1$  and  $f$  is Lipschitz, then  $f(A) - f(B) \in \mathbf{S}_\Omega$ , i.e.,  $\sum_{j=0}^n s_j(f(A) - f(B)) \leq \text{const} \log(2 + n)$ . We also obtain more general results about the behavior of double operator integrals of the form  $Q = \iint (f(x) - f(y))(x - y)^{-1} dE_1(x) T dE_2(y)$ , where  $E_1$  and  $E_2$  are spectral measures. We show that if  $T \in \mathbf{S}_1$ , then  $Q \in \mathbf{S}_\Omega$  and if  $\text{rank } T = 1$ , then  $Q \in \mathbf{S}_{1,\infty}$ . Finally, if  $T$  belongs to the Matsaev ideal  $\mathbf{S}_\omega$ , then  $Q$  is a compact operator.

## Résumé.

**Fonctions lipschitziennes d'opérateurs perturbés.** Nous démontrons que si  $f$  est une fonction lipschitzienne,  $A$  et  $B$  des opérateurs autoadjoints tels que  $\text{rank}(A - B) = 1$ , alors  $f(A) - f(B) \in \mathbf{S}_{1,\infty}$ , c'est-à-dire  $s_j(A - B) \leq \text{const}(1 + j)^{-1}$ . Si  $A - B$  est dans la classe  $\mathbf{S}_1$  des opérateurs à trace, nous montrons que  $f(A) - f(B) \in \mathbf{S}_\Omega$ , c'est-à-dire  $\sum_{j=0}^n s_j(f(A) - f(B)) \leq \text{const} \log(2 + n)$ . Plus généralement, pour une fonction lipschitzienne  $f$  et pour des mesures spectrales  $E_1$  et  $E_2$ , considérons l'intégrale double opératorielle  $Q = \iint (f(x) - f(y))(x - y)^{-1} dE_1(x) T dE_2(y)$ . Nous montrons que si  $T \in \mathbf{S}_1$ , alors  $Q \in \mathbf{S}_\Omega$  et si  $\text{rank } T = 1$ , alors  $Q \in \mathbf{S}_{1,\infty}$ . Finalement, si  $T$  appartient à l'idéal de Matsaev  $\mathbf{S}_\omega$ , alors  $Q$  est un opérateur compact.

## Version française abrégée

Dans cette note nous considérons les propriétés de  $f(A) - f(B)$ , où  $f$  est une fonction lipschitzienne sur la droite réelle  $\mathbb{R}$ ,  $A$  et  $B$  sont des opérateurs autoadjoints (pas nécessairement bornés) dont la différence  $A - B$  est "petite". Il est bien connu que si  $A - B$  appartient à l'espace  $\mathbf{S}_1$  des opérateurs nucléaires, l'opérateur  $f(A) - f(B)$  n'appartient pas nécessairement à  $\mathbf{S}_1$ .

Nous démontrons que si  $A - B \in \mathbf{S}_1$  et  $f$  est une fonction lipschitzienne, alors  $f(A) - f(B)$  appartient à l'idéal  $\mathbf{S}_\Omega$  défini comme l'ensemble d'opérateurs  $T$  dont les nombres singuliers  $s_j(T)$  satisfont à l'inégalité

$$\sum_{j=0}^n s_j(T) \leq \text{const} \log(2 + n), \quad n \geq 0.$$

Pour démontrer ce résultat nous utilisons la formule de Birman et Solomyak

$$f(A) - f(B) = \iint \frac{f(x) - f(y)}{x - y} dE_A(x)(A - B) dE_B(y),$$

où  $E_A$  et  $E_B$  sont les mesures spectrales des opérateurs  $A$  et  $B$  (la théorie des intégrales doubles opératorielles est développée dans les travaux [2], [3] et [4] de Birman et Solomyak). Nous établissons un résultat plus général: si  $f$  est une fonction lipschitzienne,  $E_1$  et  $E_2$  des mesures spectrales et  $T$  un opérateur de la classe  $\mathbf{S}_1$ , alors

$$\iint \frac{f(x) - f(y)}{x - y} dE_1(x) T dE_2(y) \in \mathbf{S}_\Omega.$$

Nous pouvons améliorer les résultats ci-dessus dans le cas  $\text{rank } T = 1$ . En réalité, dans ces cas

$$\iint \frac{f(x) - f(y)}{x - y} dE_1(x) T dE_2(y) \in \mathbf{S}_{1,\infty} \stackrel{\text{def}}{=} \left\{ T : \|T\|_{\mathbf{S}_{1,\infty}} \stackrel{\text{def}}{=} \sup_{j \geq 0} s_j(T)(1 + j) < \infty \right\}.$$

Ce fait implique que si  $A$  et  $B$  sont des opérateurs autoadjoints tels que  $\text{rank}(A - B) = 1$ , alors  $f(A) - f(B) \in \mathbf{S}_{1,\infty}$ .

En utilisant des arguments de dualité on peut montrer que si  $T$  appartient à l'idéal de Matsaev  $\mathbf{S}_\omega$ , c'est-à-dire

$$\sum_{j \geq 0} \frac{s_j(T)}{1+j} < \infty,$$

alors  $\iint (f(x) - f(y))(x - y)^{-1} dE_1(x) T dE_2(y)$  est un opérateur compact. En particulier, si  $A$  et  $B$  sont des opérateurs autoadjoints tels que  $A - B \in \mathbf{S}_\omega$ , alors  $f(A) - f(B)$  est un opérateur compact.

Pour établir les résultats ci-dessus nous montrons que si  $\mu$  et  $\nu$  sont des mesures boréliennes finies sur  $\mathbb{R}$ ,  $\varphi \in L^2(\mu)$ ,  $\psi \in L^2(\nu)$ ,

$$k(x, y) = \varphi(x) \frac{f(x) - f(y)}{x - y} \psi(y), \quad x, y \in \mathbb{R},$$

et si  $\mathcal{I}_k : L^2(\nu) \rightarrow L^2(\mu)$  est l'opérateur intégral défini par  $(\mathcal{I}_k g)(x) = \int k(x, y) g(y) d\nu(y)$ , alors

$$\sup_{j \geq 0} (1+j) s_j(\mathcal{I}_k) \leq \text{const } \|f\|_{\text{Lip}} \|f\|_{L^2(\mu)} \|\psi\|_{L^2(\nu)}.$$

En utilisant des arguments d'interpolation on peut démontrer que si  $T$  appartient à la classe de Schatten-von Neumann  $\mathbf{S}_p$ ,  $1 \leq p < \infty$ , et  $\varepsilon > 0$ , alors

$$\iint (f(x) - f(y))(x - y)^{-1} dE_1(x) T dE_2(y) \in \mathbf{S}_{p+\varepsilon}.$$

En particulier, si  $A$  et  $B$  sont des opérateurs autoadjoints tels que  $A - B \in \mathbf{S}_p$ , alors  $f(A) - f(B) \in \mathbf{S}_{p+\varepsilon}$ .

La question de savoir si la condition  $T \in \mathbf{S}_1$  implique que

$$\iint (f(x) - f(y))(x - y)^{-1} dE_1(x) T dE_2(y) \in \mathbf{S}_{1,\infty}$$

est toujours ouverte. Une réponse positive impliquerait que, dans le cas  $1 < p < \infty$ , on a  $f(A) - f(B) \in \mathbf{S}_p$  pour toute paire d'opérateurs autoadjoints  $A, B$  dont la différence  $A - B$  appartient à  $\mathbf{S}_p$ .

Finalement nous voudrions signaler qu'on peut obtenir des résultats similaires pour les fonctions d'opérateurs unitaires et pour les fonctions de contractions.

## 1. Introduction

In this note we study the behavior of Lipschitz functions of perturbed operators. It is well known that if  $f \in \text{Lip}$ , i.e.,  $f$  is a Lipschitz function and  $A$  and  $B$  are self-adjoint operators with difference in the trace class  $\mathbf{S}_1$ , then  $f(A) - f(B)$  does not have to belong to  $\mathbf{S}_1$ . The first example of such  $f$ ,  $A$ , and  $B$  was constructed in [5]. Later in [7] a necessary condition on  $f$  was found ( $f$  must be locally in the Besov space  $B_1^1$ ) under which the condition  $f(A) - f(B) \in \mathbf{S}_1$  implies that  $f(A) - f(B) \in \mathbf{S}_1$ . That necessary condition also implies that the condition  $f \in \text{Lip}$  is not sufficient.

On the other hand, Birman and Solomyak showed in [4] that if  $A - B$  belongs to the Hilbert-Schmidt class  $\mathbf{S}_2$ , then  $f(A) - f(B) \in \mathbf{S}_2$  and  $\|f(A) - f(B)\|_{\mathbf{S}_2} \leq \|f\|_{\text{Lip}} \|A - B\|_{\mathbf{S}_2}$ , where  $\|f\|_{\text{Lip}} \stackrel{\text{def}}{=} \sup_{x \neq y} |f(x) - f(y)| \cdot |x - y|^{-1}$ . Moreover, it was shown in [4] that in this case  $f(A) - f(B)$  can be expressed in terms of the following double operator integral

$$f(A) - f(B) = \iint \frac{f(x) - f(y)}{x - y} dE_A(x) (A - B) dE_B(y). \quad (1)$$

where  $E_A$  and  $E_B$  are the spectral measures of  $A$  and  $B$ . We refer the reader to [2], [3], and [4] for the beautiful theory of double operator integrals. Note that the divided difference  $(f(x) - f(y))/(x - y)$  is not defined on the diagonal. Throughout this note we assume that it is zero on the diagonal.

In this note we study properties of the operators  $f(A) - f(B)$  for (not necessarily bounded) self-adjoint operators  $A$  and  $B$  such that  $A - B$  has rank one or  $A - B \in \mathbf{S}_1$ . Actually, we consider more general operators of the form

$$\mathcal{I}_{E_1, E_2}(f, T) \stackrel{\text{def}}{=} \iint \frac{f(x) - f(y)}{x - y} dE_1(x) T dE_2(y), \quad (2)$$

where  $E_1$  and  $E_2$  are Borel spectral measures on  $\mathbb{R}$  and  $\text{rank } T = 1$  or  $T \in \mathbf{S}_1$ . Duality arguments also allow us to study double operator integrals (2) in the case when  $T$  belongs to the *Matsaev ideal*  $\mathbf{S}_\omega$ .

Recall the definitions of the following operator ideals:

$$\begin{aligned} \mathbf{S}_{1, \infty} &\stackrel{\text{def}}{=} \left\{ T : \|T\|_{\mathbf{S}_{1, \infty}} \stackrel{\text{def}}{=} \sup_{j \geq 0} s_j(T)(1 + j) < \infty \right\}, \\ \mathbf{S}_\Omega &\stackrel{\text{def}}{=} \left\{ T : \|T\|_{\mathbf{S}_\Omega} \stackrel{\text{def}}{=} (\log(2 + n))^{-1} \sum_{j=0}^n s_j(T) < \infty \right\}, \end{aligned}$$

and

$$\mathbf{S}_\omega \stackrel{\text{def}}{=} \left\{ T : \|T\|_{\mathbf{S}_\omega} \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} \frac{s_j(T)}{1 + j} < \infty \right\}.$$

It is well known that  $\mathbf{S}_{1, \infty}$  is not a Banach space and its Banach hull coincides with  $\mathbf{S}_\Omega$ . Also recall that the dual space to  $\mathbf{S}_\omega$  can be identified in a natural way with  $\mathbf{S}_\Omega$ .

Note that the recent paper [1] contains results on properties of  $f(A) - f(B)$  for  $f$  in the Hölder class  $\Lambda_\alpha$ ,  $0 < \alpha < 1$ , and self-adjoint operators  $A$  and  $B$  with  $A - B$  in Schatten–von Neuman classes  $S_p$ .

## 2. Main results

**Theorem 2.1.** *Let  $f \in \text{Lip}$  and let  $E_1$  and  $E_2$  be Borel spectral measures on  $\mathbb{R}$ . If  $\text{rank } T = 1$ , then  $\mathcal{I}_{E_1, E_2}(f, T) \in \mathbf{S}_{1, \infty}$  and*

$$\|\mathcal{I}_{E_1, E_2}(f, T)\|_{\mathbf{S}_{1, \infty}} \leq \text{const} \|f\|_{\text{Lip}} \|T\|.$$

Theorem 2.1 immediately implies the following result.

**Theorem 2.2.** *Let  $f \in \text{Lip}$  and let  $E_1$  and  $E_2$  be Borel spectral measures on  $\mathbb{R}$ . If  $T \in \mathbf{S}_1$ , then  $\mathcal{I}_{E_1, E_2}(f, T) \in \mathbf{S}_\Omega$  and*

$$\|\mathcal{I}_{E_1, E_2}(f, T)\|_{\mathbf{S}_\Omega} \leq \text{const} \|f\|_{\text{Lip}} \|T\|_{\mathbf{S}_1}.$$

By duality, we obtain the following theorem.

**Theorem 2.3.** *Let  $f \in \text{Lip}$ , and let  $E_1$  and  $E_2$  be Borel spectral measures on  $\mathbb{R}$ . Then the transformer  $T \mapsto \mathcal{I}_{E_1, E_2}(f, T)$  defined on  $\mathbf{S}_2$  extends to a bounded linear operator from  $\mathbf{S}_\omega$  to the ideal of all compact operator and*

$$\|\mathcal{I}_{E_1, E_2}(f, T)\| \leq \text{const} \|f\|_{\text{Lip}} \|T\|_{\mathbf{S}_\omega}.$$

Using interpolation arguments, we can easily obtain from Theorem 2.2 the following fact.

**Theorem 2.4.** *Let  $f \in \text{Lip}$ , and let  $E_1$  and  $E_2$  be Borel spectral measures on  $\mathbb{R}$ . Suppose that  $1 \leq p < \infty$  and  $\varepsilon > 0$ . If  $T \in \mathbf{S}_p$ , then  $\mathcal{I}_{E_1, E_2}(f, T) \in \mathbf{S}_{p+\varepsilon}$ .*

Birman–Solomyak formula (1) allows us to deduce straightforwardly from Theorems 2.1, 2.2, and 2.3 the following theorem.

**Theorem 2.5.** *Let  $A$  and  $B$  be self-adjoint operators on Hilbert space and let  $f \in \text{Lip}$ . We have*

- (i) *if  $\text{rank}(A - B) = 1$ , then  $f(A) - f(B) \in \mathbf{S}_{1, \infty}$  and  $\|f(A) - f(B)\|_{\mathbf{S}_{1, \infty}} \leq \text{const} \|f\|_{\text{Lip}} \|A - B\|$ ;*
- (ii) *if  $A - B \in \mathbf{S}_1$ , then  $f(A) - f(B) \in \mathbf{S}_\Omega$  and  $\|f(A) - f(B)\|_{\mathbf{S}_\Omega} \leq \text{const} \|f\|_{\text{Lip}} \|A - B\|_{\mathbf{S}_1}$ ;*
- (iii) *if  $A - B \in \mathbf{S}_\omega$ , then  $f(A) - f(B)$  is compact and  $\|f(A) - f(B)\| \leq \text{const} \|f\|_{\text{Lip}} \|A - B\|_{\mathbf{S}_\omega}$ ;*
- (iv) *if  $1 \leq p < \infty$ ,  $\varepsilon > 0$ , and  $A - B \in \mathbf{S}_p$ , then  $f(A) - f(B) \in \mathbf{S}_{p+\varepsilon}$ .*

It is still unknown whether the assumption  $T \in \mathbf{S}_1$  implies that  $\mathcal{I}_{E_1, E_2}(f, T) \in \mathbf{S}_{1, \infty}$ . If this is true, then the condition  $A - B \in \mathbf{S}_p$  would imply that  $f(A) - f(B) \in \mathbf{S}_p$  for  $1 < p < \infty$ .

To prove Theorem 2.1, we obtain a weak type estimate for Schur multipliers.

For a kernel function  $k \in L^2(\mu \times \nu)$ , we define the integral operator  $\mathcal{I}_k : L^2(\nu) \rightarrow L^2(\mu)$  by

$$(\mathcal{I}_k g)(x) = \int k(x, y) g(y) d\nu(y), \quad g \in L^2(\nu).$$

As in the case of transformers from  $\mathbf{S}_1$  to  $\mathbf{S}_1$  (see [4]), Theorem 2.1 reduces to the following fact.

**Theorem 2.6.** *Let  $\mu$  and  $\nu$  be finite Borel measures on  $\mathbb{R}$ ,  $\varphi \in L^2(\mu)$ ,  $\psi \in L^2(\nu)$ . Suppose that  $f \in \text{Lip}$  and the kernel function  $k$  is defined by*

$$k(x, y) = \varphi(x) \frac{f(x) - f(y)}{x - y} \psi(y), \quad x, y \in \mathbb{R}.$$

*Then the integral operator  $\mathcal{I}_k : L^2(\nu) \rightarrow L^2(\mu)$  with kernel function  $k$  belongs to  $\mathbf{S}_{1, \infty}$  and*

$$\|\mathcal{I}_k\|_{\mathbf{S}_{1, \infty}} \leq \text{const} \|f\|_{\text{Lip}} \|\varphi\|_{L^2(\mu)} \|\psi\|_{L^2(\nu)}.$$

**Proof.** Without loss of generality we may assume that  $\|\varphi\|_{L^2(\mu)} = \|\psi\|_{L^2(\nu)} = 1$  and  $\|f\|_{\text{Lip}} = 1$ . Let us fix a positive integer  $n$ .

Given  $N > 0$ , we denote by  $P_N$  multiplication by the characteristic function of  $[-N, N]$  (we use the same notation for multiplication on  $L^2(\mu)$  and on  $L^2(\nu)$ ). Then for sufficiently large values of  $N$ ,

$$\|\mathcal{I}_k - P_N \mathcal{I}_k P_N\|_{\mathbf{S}_2} < \frac{1}{n^{1/2}}. \quad (3)$$

Clearly,  $P_N \mathcal{I}_k P_N$  is the integral operator with kernel function  $k_N$ ,  $k_N(x, y) = \chi_N(x) k(x, y) \chi_N(y)$ , where  $\chi_N = \chi_{[-N, N]}$  is the characteristic function of  $[-N, N]$ . We fix  $N > 0$ , for which (3) holds.

Consider now the points  $x_j$ ,  $1 \leq j \leq r$ , and  $y_j$ ,  $1 \leq j \leq s$ , at which  $\mu$  and  $\nu$  have point masses and

$$|\varphi(x_j)|^2 \mu\{x_j\} \geq \frac{1}{n}, \quad 1 \leq j \leq r, \quad \text{and} \quad |\psi(y_j)|^2 \nu\{y_j\} \geq \frac{1}{n}, \quad 1 \leq j \leq s. \quad (4)$$

Clearly,  $r \leq n$  and  $s \leq n$ . We define now the kernel function  $k_\#$  by

$$k_\#(x, y) = u(x) k_N(x, y) v(y), \quad x, y \in \mathbb{R},$$

where

$$u(x) \stackrel{\text{def}}{=} 1 - \chi_{\{x_1, \dots, x_r\}}(x) \quad \text{and} \quad v(y) \stackrel{\text{def}}{=} 1 - \chi_{\{y_1, \dots, y_s\}}(y).$$

Obviously, the integral operators  $\mathcal{I}_{k_N}$  and  $\mathcal{I}_{k_\#}$  coincide on a subspace of codimension at most  $r + s \leq 2n$ .

We can split now the interval  $[-N, N]$  into no more than  $n$  subintervals  $I$ ,  $I \in \mathfrak{J}$ , such that

$$\int_I |\varphi(x)|^2 u(x) d\mu(x) + \int_I |\psi(y)|^2 v(y) d\nu(y) \leq \frac{4}{n}, \quad I \in \mathfrak{J}.$$

This is certainly possible because of (4).

We have  $\mathcal{I}_{k_\#} = \mathcal{I}^{(1)} + \mathcal{I}^{(2)} + \mathcal{I}^{(3)}$ , where

$$(\mathcal{I}^{(1)} g)(x) = \int_{\mathbb{R}} \left( \sum_{I \in \mathfrak{J}} \chi_I(x) k_\#(x, y) \chi_I(y) \right) g(y) d\nu(y),$$

$$(\mathcal{I}^{(2)} g)(x) = \int_{\mathbb{R}} \left( \sum_{I, J \in \mathfrak{J}, I \neq J, |I| \geq |J|} \chi_I(x) k_\#(x, y) \chi_I(y) \right) g(y) d\nu(y),$$

and

$$(\mathcal{I}^{(3)} g)(x) = \int_{\mathbb{R}} \left( \sum_{I, J \in \mathfrak{J}, |I| < |J|} \chi_I(x) k_\#(x, y) \chi_I(y) \right) g(y) d\nu(y)$$

(we denote by  $|I|$  the length of  $I$ ). It is easy to see that  $\|\mathcal{I}^{(1)}\|_{\mathcal{S}_2} \leq 4n^{-1/2}$ . Let us estimate  $\mathcal{I}^{(2)}$ . The integral operator  $\mathcal{I}^{(3)}$  can be estimated in the same way.

Suppose that  $I, J \in \mathfrak{J}$ ,  $I \neq J$ , and  $|I| \geq |J|$ . For  $x \in I$  and  $y \in J$ , we have

$$\frac{1}{x-y} = \frac{1}{x-c(J)} + \frac{y-c(J)}{x-c(J)} \cdot \frac{1}{x-y},$$

where  $c(J)$  denotes the center of  $J$ .

Suppose that  $g \perp \bar{\psi}\chi_J$  and  $g \perp \bar{\psi}f\chi_J$ . Then  $\mathcal{I}_2 g = \mathcal{I}_{k_b} g$ , where

$$k_b(x, y) = \sum_{I, J \in \mathfrak{J}, I \neq J, |I| \geq |J|} u(x)\varphi(x)a_{IJ}(x, y)\psi(y)v(y)$$

and

$$a_{IJ}(x, y) = \chi_I(x) \frac{y-c(J)}{x-c(J)} \cdot \frac{f(x)-f(y)}{x-y} \chi_J(y).$$

Thus  $\mathcal{I}^{(2)}$  and  $\mathcal{I}_{k_b}$  coincide on a subspace of codimension at most  $2n$ .

To estimate the Hilbert–Schmidt norm of  $\mathcal{I}_{k_b}$ , we observe that

$$|a_{IJ}(x, y)| \leq \frac{|J|}{(|J| + \text{dist}(I, J))}, \quad x \in I, \quad y \in J.$$

Thus

$$\begin{aligned} \|\mathcal{I}_{k_b}\|_{\mathcal{S}_2}^2 &\leq \sum_{I, J \in \mathfrak{J}, I \neq J, |I| \geq |J|} \left( \int_I |\varphi|^2 u \, d\mu \right) \left( \int_J |\psi|^2 v \, d\nu \right) \|a_{IJ}\|_{L^\infty}^2 \\ &\leq \frac{4}{n^2} \sum_{I, J \in \mathfrak{J}, I \neq J, |I| \geq |J|} \frac{|J|^2}{(|J| + \text{dist}(I, J))^2}. \end{aligned}$$

Let us observe that for a fixed  $J \in \mathfrak{J}$ ,

$$\sum_{I \in \mathfrak{J}, I \neq J, |I| \geq |J|} \frac{|J|^2}{(|J| + \text{dist}(I, J))^2} \leq \text{const}. \quad (5)$$

Indeed, we can enumerate the intervals  $I \in \mathfrak{J}$  satisfying  $I \neq J$  and  $|I| \geq |J|$  so that the resulting intervals  $I_k$  satisfy  $\text{dist}(I_k, J) \leq \text{dist}(I_{k+1}, J)$ . Since the intervals  $I_k$  are disjoint, we have

$$\text{dist}(I_k, J) \geq \frac{k-3}{2}|J|.$$

This easily implies (5). It follows that

$$\|\mathcal{I}_{k_b}\|_{\mathcal{S}_2}^2 \leq C \frac{4}{n^2} \cdot n = \frac{4C}{n}.$$

Similarly,  $\mathcal{I}^{(3)}$  coincides on a subspace of codimension at most  $2n$  with an operator whose Hilbert–Schmidt norm is at most  $2(C/n)^{1/2}$ .

If we summarize the above, we see that  $\mathcal{I}_k$  coincides on a subspace of codimension at most  $6n$  with an operator whose Hilbert–Schmidt norm is at most  $Kn^{-1/2}$ , where  $K$  is a constant. Hence, on a subspace of codimension at most  $7n$  the operator  $\mathcal{I}_k$  coincides with an operator whose norm is at most  $K/n$ , i.e.,

$$s_{7n}(\mathcal{I}_k) \leq \frac{K}{n}, \quad n \geq 1, \quad \blacksquare$$

Note that in the case of operators on the space  $L^2(\mathbb{T})$  with respect to Lebesgue measure on the unit circle  $\mathbb{T}$ , the following related fact was obtained in [6] (see also [8]): if the derivative of  $f$  belongs to the Hardy class  $H^1$ ,  $\varphi$  and  $\psi$  belong to  $L^\infty(\mathbb{T})$ , and the kernel function  $k$  is defined by

$$k(\zeta, \tau) = \varphi(\zeta) \frac{f(\zeta) - f(\tau)}{\zeta - \tau} \psi(\tau), \quad \zeta, \tau \in \mathbb{T},$$

then the integral operator  $\mathcal{I}_k$  on  $L^2(\mathbb{T})$  belongs to  $\mathcal{S}_{1,2}$ , i.e.,  $\sum_{j \geq 0} (s_j(\mathcal{I}_k))^2 (1+j) < \infty$ .

To conclude the article, we note that similar results can be obtained for functions of unitary operators and for functions of contractions.

**Remark.** After this article had been written we have been informed by D. Potapov and F. Sukochev that they had proved the following result: if  $f$  is a Lipschitz function,  $1 < p < \infty$ , and  $A$  and  $B$  are self-adjoint operators such that  $A - B \in \mathcal{S}_p$ , then  $f(A) - f(B) \in \mathcal{S}_p$ .

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